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COMPOSITION ALGEBRAS OVER RINGS OF GENUS ZERO

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ABSTRACT. The theory of composition algebras over locally ringed spaces and some basic results from algebraic geometry are used to characterize composition algebras over open dense subschemes of curves of genus zero.

INTRODUCTION

While the theory of composition algebras over fields is relatively well developed (see, for instance, the classical results from Albert [A], Jacobson [J], and van der Blij and Springer [BS]) composition algebras over (unital, commutative, associative) rings are harder to understand. Some general results are due to McCrimmon [M] and Petersson [P1]. The theory of composition algebras over locally ringed spaces by Petersson [P1] yields results for composition algebras over affine schemes which can be rephrased for composition algebras over rings. In particular, this leads to a generalization of the classical Cayley-Dickson doubling (cf. [A]) for composition algebras over rings and to a characterization of split composition algebras over rings. However, while any composition algebra of rank $r > 2$ over a field can be obtained from the Cayley-Dickson doubling process, this generally does not hold for composition algebras over rings.

In order to obtain more detailed results, the only promising line of attack seems to be to study certain classes of rings. For a principal ideal domain R it is known that any composition algebra which contains zero divisors splits and is isomorphic to $R \oplus R$, or $\mathrm{Mat}_2(R)$ (the 2-by-2 matrices over R), or Zorn's algebra of vector matrices $\mathrm{Zor}\, R$ (cf. [P1], 3.6). The proof goes back to [BS]. Knus, Parimala and Sridharan [KPS] investigated composition algebras over polynomial rings in n variables over fields k of $\mathrm{char}\, k \neq 2$.

Nondegenerate symmetric bilinear spaces over open dense subschemes of the projective line were studied by Harder and Knebusch ([Kn] and [L], VI.3.13). Harder's classical result on forms over polynomial rings was adapted in [P1] to prove, in particular, that a composition algebra over $k[t]$ is defined over k for any field k of characteristic not two.

This paper takes a slightly more general approach and investigates composition algebras over open dense subschemes of curves of genus zero using some algebraic geometry as well as the classification of composition algebras over such curves.

The basic terminology needed is given in section 1, the remainder can be found in [P1], [P2] and [G]. The main results are stated in section 3. For a ring R where

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Spec R is an open dense subscheme of a curve X' of genus zero, composition algebras without zero divisors are characterized. The ramification behaviour of these composition algebras over the function field of X' is investigated using the theory of valuations on composition algebras over fields complete under a discrete valuation (cf. Küting [Kü], [P1], or [P2]). A composition algebra C without zero divisors over R is defined over the base field k of X' if and only if it is unramified at all closed points of X' (2.7). It can be proved that certain composition algebras contain composition subalgebras of half rank defined over k and that certain octonion algebras contain tori defined over k (3.6). In particular, we get examples of rings where any composition algebra without zero divisors can be realized by a generalized Cayley-Dickson doubling (3.10). Moreover, we obtain rings where any octonion algebra contains a torus defined over k (3.11). Also, if the respective composition algebra itself is not defined over k, these subalgebras are uniquely determined up to isomorphism (3.9). Finally, section 4 deals with composition algebras having zero divisors. In particular, it is shown that every composition algebra with zero divisors splits, in case the curve of genus zero considered is rational (4.3), and that in this case Zorn's vector matrix algebra is the only octonion algebra with zero divisors.

In forthcoming papers the results proven here will be used to enumerate all composition algebras both over a ring of fractions

$$R = \left\{ \frac{g(t)}{f(t)^j} \in k(t) \mid j \geq 0,\ g(t) \in k[t] \quad \text{with } \deg g \leq 2j \right\}$$

for a monic irreducible polynomial $f(t) \in k[t]$ of degree two, and over the ring $k[t, \sqrt{at^2+b}]$ (cf. (3.10)), and to partly classify them.

The paper uses the standard terminology of algebraic geometry from Hartshorne [H], as well as the standard terminology of quadratic forms from Scharlau [S]. Special cases of the results mentioned here have been announced without proof in [Pu1]. The main results of this article first appeared in the author's doctoral thesis [Pu2].

1. Preliminaries

1.1. Let R be a commutative associative ring with a unit element. An R-module M is said to have *full support* if $M_P \neq 0$ for all $P \in \operatorname{Spec} R$. In this paper the term "R-algebra" refers to unital nonassociative algebras which are finitely generated projective of rank > 0 as R-modules. An R-algebra C is said to be a *composition algebra* in case it has full support and admits a quadratic form $N : C \to R$ satisfying the following two conditions:

(i) Its induced symmetric bilinear form $N\colon C \times C \to R$, $N(u,v) := N(u+v) - N(u) - N(v)$ is nondegenerate, i.e. it determines an R-module isomorphism $C \xrightarrow{\sim} \check{C} = \operatorname{Hom}_R(C,R)$,

(ii) N permits composition, i.e. $N(uv) = N(u)N(v)$ for all $u, v \in C$.

An R-algebra C is called *quadratic* in case there exists a quadratic form $N : C \to R$ such that $N(1_C) = 1$ and $u^2 - N(1_C, u)u + N(u)1_C = 0$ for all $u \in C$. The form N is uniquely determined and called the *norm* of the quadratic R-algebra C. An R-algebra C is called *alternative* if its associator $[u,v,w] = (uv)w - u(vw)$ is alternating. Given a quadratic alternative R-algebra C a *composition subalgebra* of C is defined to be a unital subalgebra which is a composition algebra. Composition algebras over rings are quadratic alternative algebras. More precisely, a quadratic form N of the composition algebra satisfying conditions (i) and (ii) above agrees with the norm of the quadratic algebra C and therefore is uniquely determined. N is called the *norm* of the composition algebra C and sometimes also denoted by N_C. The map $^* : C \to C$, $u^* := N_C(1_C, u)1_C - u$, which is an algebra involution, is called the *canonical involution of* C. Furthermore, a quadratic alternative algebra with full support is a composition algebra if and only if its norm is nondegenerate ([M], 4.6). Composition algebras over rings only exist in ranks 1, 2, 4, or 8. A composition algebra of rank 2 (resp. 4, resp. 8) is called *torus* (resp. *quaternion algebra*, resp. *octonion algebra*). Composition algebras are invariant under base change.

The following is the result of a straightforward computation.

1.2. Lemma. *Let R be an integral domain, $K = \operatorname{Quot}(R)$ its quotient field and let C be a composition algebra over R with norm N. Then the following are equivalent.*
(i) *C contains no zero divisors.*
(ii) *$C \otimes K$ contains no zero divisors.*
(iii) *N is anisotropic.*

A composition algebra over R is called *split* if it contains a composition subalgebra isomorphic to $R \oplus R$, which is a torus with (hyperbolic) norm $(a, b) \to ab$. Every split composition algebra has zero divisors.

1.3. Let X be a locally ringed space with structure sheaf $\mathcal{O}_X$. For $P \in X$, $\mathcal{O}_{P,X}$ denotes the local ring of $\mathcal{O}_X$ at P, m_P the maximal ideal of $\mathcal{O}_{P,X}$ and $\kappa(P) = \mathcal{O}_{P,X}/m_P$ the corresponding residue class field. For an $\mathcal{O}_X$-module $\mathcal{F}$ the stalk of $\mathcal{F}$ at P is denoted by $\mathcal{F}_P$. A *quadratic form* $Q : \mathcal{F} \to \mathcal{O}_X$ is a family

$$(Q(U))_{U \subset X, U \text{ open}}$$

of quadratic forms $Q(U) : \mathcal{F}(U) \to \mathcal{O}_X(U)$ which are compatible with the restriction morphisms. A *bilinear form* $B : \mathcal{F} \times \mathcal{F} \to \mathcal{O}_X$ is defined analogously. B is called *nondegenerate* if it induces an isomorphism $\mathcal{F} \xrightarrow{\sim} \check{\mathcal{F}} \cong \operatorname{Hom}_X(\mathcal{F}, \mathcal{O}_X)$. In case $\mathcal{F}$ is locally free of finite rank this is equivalent to $B_P : \mathcal{F}_P \times \mathcal{F}_P \to \mathcal{O}_P$, X being nondegenerate for all $P \in X$. Note that even for a nondegenerate bilinear form B the sections $B(U) : \mathcal{F}(U) \times \mathcal{F}(U) \to \mathcal{O}_X(U)$ may be degenerate. A quadratic form $Q : \mathcal{F} \to \mathcal{O}_X$ canonically induces a symmetric bilinear form $Q : \mathcal{F} \times \mathcal{F} \to \mathcal{O}_X$.

1.4. Definition ([P1], 1.6)**.** An $\mathcal{O}_X$-algebra $\mathcal{C}$ is said to be a *composition algebra* over X in case $\mathcal{C}_P \neq 0$ for all $P \in X$ and there exists a quadratic form $N : \mathcal{C} \to \mathcal{O}_X$ satisfying:
(i) Its induced symmetric bilinear form is nondegenerate,
(ii) $N(uv) = N(u)N(v)$ for all sections u, v in $\mathcal{C}$.

In this context "$\mathcal{O}_X$-algebra" always refers to unital nonassociative algebras which are locally free of finite rank as $\mathcal{O}_X$-modules. Quadratic and alternative algebras over X are defined likewise. Again, composition algebras only exist in ranks 1, 2, 4 or 8 and are called *tori*, *quaternion algebras or octonion algebras*, respectively. In accordance with the definition for composition algebras over rings a composition algebra over X is called *split*, if it contains a composition subalgebra isomorphic to $\mathcal{O}_X \oplus \mathcal{O}_X$ ([P1], 1.7, 1.8). Let X be an R-scheme and $\tau : X \to \operatorname{Spec} R$ be its structure morphism, then a composition algebra $\mathcal{C}$ over X is *defined over* R in case there exists a composition algebra C over R with $\mathcal{C} = \tau^* C \cong C \otimes \mathcal{O}_X$.

There is a canonical equivalence between the category of composition algebras over the affine scheme $Z := \operatorname{Spec} R$ and the category of composition algebras over

R. It is given by the global section functor

$$\mathcal{C} \to \Gamma(Z, \mathcal{C})$$

and the functor

$$C \to \widetilde{C}$$

(cf. [P1], 1.9). With the help of this equivalence the generalized Cayley-Dickson doubling for composition algebras over locally ringed spaces by Petersson ([P1], 2.3, 2.4, 2.5) can be formulated for composition algebras over rings, which is briefly done in (1.5) for the convenience of the reader. Further properties of composition algebras over X along with terminology and concepts not explained here can be found in [P1].

1.5. Let D be a composition algebra of rank ≤ 4 over R, let $\mathrm{Pic}_r\, D$ denote the (pointed) set of isomorphism classes of projective right D-modules of rank one. Moreover, let $Z = \mathrm{Spec}\, R$ and view the group of units $D^\times$ of D as a group scheme. Then $\mathrm{Pic}_r\, D = \check{H}^1(Z, D^\times)$ as pointed sets in the sense of noncommutative Čech-cohomology ([Mi], III.4.6). The homomorphism $N_D : D^\times \to \mathbf{G}_m$ canonically induces a homomorphism $N_D : \mathrm{Pic}_r\, D \to \mathrm{Pic}\, R$ of pointed sets. Given a projective right D-module P of rank one, it is said to have *norm one* if $N_D(P) \cong R$. In case P has norm one, there exists a nondegenerate quadratic form $N : P \to R$ satisfying $N(w \cdot u) = N(w)N_D(u)$ for $w \in P$, $u \in D$, where $\cdot$ denotes the right D-module structure of P. N is uniquely determined up to a factor $\mu \in R^\times$ and called a *norm* on P. Furthermore, N determines a unique R-bilinear map $P \times P \to D$, written multiplicatively and satisfying $(w \cdot u)(w \cdot v) = N(w)v^* u$ for $w \in P$, $u, v \in D$. Now the R-module

$$\mathrm{Cay}(D, P, N) := D \oplus P$$

becomes a composition algebra under the multiplication

$$(u, w)(u', w') = (uu' + ww', w' \cdot u + w \cdot u'^*)$$

with norm $N_{\mathrm{Cay}(D,P,N)} = N_D \oplus (-N)$.

Conversely, given a composition algebra C with $\mathrm{rank}\, C = 2 \cdot \mathrm{rank}\, D$ containing D as a subalgebra, there are P, N as above such that $C \cong \mathrm{Cay}(D, P, N)$.

Considering the free right D-module $D \in \mathrm{Pic}_r\, D$ itself, D has norm one and any norm on D is similar to N_D. It turns out that in this case we get the *classical* Cayley-Dickson doubling process $\mathrm{Cay}(D, \mu) := \mathrm{Cay}(D, D, \mu N_D)$ (cf., for instance, [P1], 2.1, 2.2) due to Albert [A]. Here, the abbreviation $\mathrm{Cay}(D, \mu, \eta) := \mathrm{Cay}(\mathrm{Cay}(D, \mu), \eta)$ is also used for $\mathrm{rank}\, D < 4$.

1.6. From now on let Y denote a *curve* over an arbitrary base field k, that is, a geometrically integral, complete, smooth scheme of dimension one and of finite type over k. Here, "point" without further specification always refers to a closed point, whereas ξ denotes the generic point of Y. The *function field* of Y is $K := \kappa(Y) := \mathcal{O}_{\xi, P}$. The base field k is algebraically closed in K. A point $P \in Y$ is called *k-rational* if its *degree* $\deg P := [\kappa(P) \colon k] = 1$. For a K-algebra C the constant sheaf of C over Y is denoted by $\underline{C}$. Note that $\Gamma(U, \underline{C}) = C$ for all non-empty open sets $U \subset Y$.

1.7. Let R be a Dedekind domain, $K = \mathrm{Quot}(R)$ its quotient field and V a finite-dimensional K-vector space. Following Reiner [R], p. 44, a finitely generated R-module M is said to be a (full) *R-lattice* in V in case M contains a K-basis of V, that is $KM = V$. Any such lattice is a projective R-module ([R], 4.13).

An *$\mathcal{O}_Y$-lattice* $\mathcal{M}$ in V is a locally free $\mathcal{O}_Y$-submodule of $\underline{V}$ of rank $n = \dim_K V$.

For an $\mathcal{O}_Y$-lattice $\mathcal{M}$ in V the stalk $\mathcal{M}_P$ is an $\mathcal{O}_{P,Y}$-lattice in V for all $P \in X$. Given $\mathcal{O}_{P_i,Y}$-lattices $L(P_i)$ in V for finitely many $P_1, \ldots, P_s \in Y$ and an $\mathcal{O}_Y(U)$-lattice $L(U)$ for $U = Y - \{P_1, \ldots, P_s\}$, there exists an $\mathcal{O}_Y$-lattice $\mathcal{L}$ in V such that $\mathcal{L}|_U = \widetilde{L(U)}$ and $\mathcal{L}_{P_i} = L(P_i)$ for $1 \leq i \leq s$ (1.8, cf. [G], 1.8).

The *genus* of a curve Y is $g_Y := h^1(Y, \mathcal{O}_Y)$, while $\chi(\mathcal{M}) = h^0(Y, \mathcal{M}) - h^1(Y, \mathcal{M})$ denotes the *Euler characteristic* of $\mathcal{M}$ ([G], 1.15).

For $\mathcal{O}_Y$-lattices $\mathcal{M}, \mathcal{N}$ in V, gluing the $\mathcal{O}_Y(U_i)$-order ideals $[\mathcal{M}(U_i) : \mathcal{N}(U_i)]$ (for the definition cf. [G], 1.16) for an open affine covering $\{U_i\}$ of Y yields an $\mathcal{O}_Y$-lattice in V denoted by $[\mathcal{M} : \mathcal{N}]$. Now, $[\mathcal{M} : \mathcal{N}]_P = [\mathcal{M}_P : \mathcal{N}_P]$ for all $P \in Y$. Furthermore, $\deg_{\mathcal{N}} \mathcal{M} = \sum_{P \in Y} v_P([\mathcal{M}_P : \mathcal{N}_P]) \deg P$ is said to be the *degree* of $\mathcal{M}$ with respect to $\mathcal{N}$ ([G], 1.18). Let (V, b) be a nondegenerate symmetric bilinear space and $\mathcal{M}$ an $\mathcal{O}_Y$-lattice in V. Gluing the $\mathcal{O}_Y(U_i)$-lattices $\mathcal{M}(U_i)^{\#} = \{v \in V \mid b(v, \mathcal{M}(U_i)) \subset \mathcal{O}_Y(U_i)\}$, where $\{U_i\}$ again is an open affine covering of Y, yields an $\mathcal{O}_Y$-lattice in V (the *dual lattice*) denoted by $\mathcal{M}^{\#}$ ([G], 1.20).

1.8. Let R be a Dedekind domain, $K = \mathrm{Quot}(R)$ its quotient field and A a finite dimensional alternative K-algebra. Then an R-lattice in A is said to be an *R-order*, if it is multiplicatively closed and contains the unit element of A. It is called *maximal* in case $M \subset M'$ implies $M = M'$ for all R-orders M' in A.

For a composition algebra C over $K = \kappa(Y)$, an $\mathcal{O}_Y$-lattice $\mathcal{M}$ in C is called an *$\mathcal{O}_Y$-order* in C if $\mathcal{M} \subset \underline{C}$ is an $\mathcal{O}_Y$-algebra and $\mathcal{M}_P \subset C$ an $\mathcal{O}_{P,Y}$-order for all $P \in Y$. An $\mathcal{O}_Y$-order $\mathcal{M}$ in C is called *maximal* if $\mathcal{M} \subset \mathcal{M}' \subset \underline{C}$ implies $\mathcal{M} = \mathcal{M}'$ for any $\mathcal{O}_Y$-order $\mathcal{M}'$ in C. For an $\mathcal{O}_Y$-order $\mathcal{M}$ in C the following are equivalent:

(i) $\mathcal{M}$ is maximal,

(ii) $\mathcal{M}_P$ is a maximal $\mathcal{O}_{P,Y}$-order in C for all $P \in Y$,

(iii) $\mathcal{M}(U)$ is a maximal $\mathcal{O}_Y(U)$-order in C for all open dense affine subsets U of Y.

There always exists a maximal $\mathcal{O}_Y$-order in C. The proof is similar to the one for associative algebras (cf. [R], Chap. 2).

1.9. $\mathcal{O}_{P,Y}$ is a discrete valuation ring for any $P \in Y$. It corresponds with a valuation v_P on $K = \kappa(Y)$ which is trivial on k. $\widehat{K}_P$ denotes the completion of K with respect to this valuation and $\widehat{\mathcal{O}}_{P,Y}$ the respective valuation ring of $\widehat{K}_P$. A composition algebra C over K is said to be *unramified at* P if $\widehat{C}_P = C \otimes \widehat{K}_P$ either splits or is an unramified composition division algebra over $\widehat{K}_P$. C is said to be (*separably*) *ramified at* P if $\widehat{C}_P$ is a (separably) ramified division algebra (cf. [Kü] or [P1], 6.2). C is unramified at P if and only if $\widehat{C}_P$ contains a selfdual $\widehat{\mathcal{O}}_{P,Y}$-order ([P1], 6.3). For the definition of unramified as well as (separably) ramified composition algebras over fields complete under a discrete valuation the reader is referred to [P1], 6.2.

2. Composition algebras without zero divisors

2.1. Fix an arbitrary base field k and let X' denote a curve of genus zero over k. Hence, X' is either isomorphic to $\mathbf{P}^1_k = \mathrm{Proj}\, k[x_0, x_1]$ and called *rational*, or it is not isomorphic to $\mathbf{P}^1_k$ but becomes isomorphic to the projective line after passing to the algebraic closure or to an appropriate quadratic extension of k (so $X' \times_k k' = \mathbf{P}^1_k$ for such a field extension k' of k), in which case X' is said to be *nonrational*. X' contains rational points if and only if it is rational. X' always contains points of degree at most two (cf. [T]). For rational X' the function field is $K = k(t)$, for nonrational X' it is a quadratic extension of a purely transcendental field extension of k of transcendence degree one. There is a one-to-one correspondence between (nonrational) curves of genus zero and quaternion (division) algebras over k ([T], 5.4 or [Wi]). Moreover, for a nonrational curve X' there exists an indecomposable locally free $\mathcal{O}_{X'}$-module $\mathcal{E}$ of rank two which is unique up to multiplication with a unique invertible sheaf. The quaternion algebra associated with X' then is $D = \mathrm{End}_{X'}(\mathcal{E})$ ([P1], 4.3 or [T], 5.4). D represents the element in the Brauer group $\mathrm{Br}(k)$ which does not equal the neutral element $\mathrm{Mat}_2(k)$ and splits over K, that is $D \otimes_k K \cong \mathrm{Mat}_2(K)$ ([T], 5.4). Using 1.2 one can easily show the following.

2.2. Proposition. *Let X' be a nonrational curve of genus zero and R an integral domain such that $\mathrm{Quot}(R) = \kappa(X') = K$, and $k \subset R$. For a quaternion algebra C_0 over k the algebra $C_0 \otimes R$ has zero divisors if and only if $C_0 \cong \mathrm{Mat}_2(k)$ or $C_0 \cong D$.*

One of the main results of Petersson [P1] is the classification of composition algebras over curves of genus zero.

2.3. Theorem ([P1], 4.4)**.** *Let $\mathcal{C}$ be a composition algebra over X'. Then one of the following holds.*

(i) *$\mathcal{C}$ is defined over k.*
(ii) *$\mathcal{C}$ is a split quaternion algebra.*
(iii) *$\mathcal{C}$ is a split octonion algebra.*
(iv) *X' is nonrational and $\mathcal{C} \cong \mathrm{Cay}(D \otimes \mathcal{O}_{X'}, \mathcal{P}, N)$, where D is the quaternion division algebra over k associated with X', $\mathcal{P} \in \mathrm{Pic}_r(D \otimes \mathcal{O}_{X'})$ is of norm one, and N is a norm on $\mathcal{P}$.*

As in the category of composition algebras over rings, $\mathrm{Pic}_r(\mathcal{D})$ denotes the (pointed) set of isomorphism classes of locally free right $\mathcal{D}$-modules of rank one, where $\mathcal{D}$ is an associative composition algebra over a locally ringed space. For the notions not explained here, for instance, the definition of $\mathrm{Cay}(\mathcal{D}, \mathcal{P}, N)$, the reader is referred to [P1], Sec. 2. The following is a well-known fact from algebraic geometry.

2.4. Proposition. *Let Y be a curve over k and $P_0, \dots, P_n \in Y$. Then*

$$Y - \{P_0, \dots, P_n\} \cong \mathrm{Spec}\, R$$

and R is a Dedekind domain with quotient field $\mathrm{Quot}(R) = \kappa(Y)$, that is, any proper open subset of Y is affine.

Using the equivalence between the category of composition algebras over $\mathrm{Spec}\, R$ and the category of composition algebras over R (1.4) it is possible to investigate composition algebras over these Dedekind domains.

For every curve Y over k, k an arbitrary base field, there is a connection between composition algebras over

$$X := \mathrm{Spec}\, R = Y - \{P_0, \dots, P_n\}$$

and composition algebras over

$$Z := \mathrm{Spec}\, S = Y - \{P_{i_0}, \dots, P_{i_m}\}$$

with $P_{i_0}, \dots, P_{i_m} \in \{P_0, \dots, P_n\}$. Because of $\operatorname{Spec} R \subset \operatorname{Spec} S$ consider S to be a subring of R.

2.5. *Remark.* For a ring R with $\operatorname{Spec} R = Y - \{P_0, \dots, P_n\}$ the composition algebra $C' := C \otimes_R K$, $K := \kappa(Y)$ is unramified at all $P \in \operatorname{Spec} R$ for any composition algebra C over R. This follows from the fact that $\widetilde{C}_P$ is a selfdual $\mathcal{O}_{P,Y}$-order in C' for all $P \in \operatorname{Spec} R$. So the topological closure in $C \otimes_R \widehat{K}_P$ is a selfdual $\widehat{\mathcal{O}}_{P,Y}$-order implying that C' is unramified at all $P \in \operatorname{Spec} R$ (1.9).

2.6. Proposition. (i) *A composition algebra C over R is defined over S (that is $C \cong C_1 \otimes_S R$ for a composition algebra C_1 over S) if and only if $C' := C \otimes_R K$ is unramified at all $P \notin \{P_{i_0}, \dots, P_{i_m}\}$.*

(ii) *There exists a composition algebra C over R such that $C \otimes_R K$ ramifies exactly at $P_{i_0}, \dots, P_{i_m}$ if and only if there exists a composition algebra C_1 over S such that $C_1 \otimes_S K$ ramifies exactly at $P_{i_0}, \dots, P_{i_m}$.*

Proof. We may assume $P_{i_l} = P_l$ for $0 \leq l \leq m$, so $Y - \{P_0, \dots, P_m\} = \operatorname{Spec} S$.

(i) If C is a composition algebra over R with C' unramified at $P_{m+1}, \dots, P_n$ and $\mathcal{C}'$ a maximal $\mathcal{O}_Y$-order in C' satisfying $\mathcal{C}'|_X = \widetilde{C}$, then $\mathcal{C}'_{P_i}$ is a composition algebra over $\mathcal{O}_{P_i,Y}$ for $m+1 \leq i \leq n$. Thus $\mathcal{C}'|_Z$ is a composition algebra, because $Z = X \cup \{P_{m+1}, \dots, P_n\}$. There exists a composition algebra C_1 over S such that $\widetilde{C}_1 = \mathcal{C}'|_Z$. Let $\iota : X \to Z$ denote the canonical inclusion, then $\widetilde{C} = \mathcal{C}'|_X = \iota^* C_1 = C_1 \otimes_{\mathcal{O}_Z} \mathcal{O}_X$ implying $C \cong C_1 \otimes_S R$.

Conversely, $C = C_1 \otimes_S R$ for a composition algebra C_1 over S yields that $C' \cong C_1 \otimes_S K$ is unramified at all $P \in Z = \operatorname{Spec} S$ (2.5).

(ii) For a composition algebra C_1 over S with $C_1 \otimes_S K$ ramifying exactly at $P_0, \dots, P_n$ define $C := C_1 \otimes_S R$.

Conversely, if C is a composition algebra over R such that $C \otimes_R K$ ramifies exactly at $P_0, \dots, P_n$, $C \cong C_1 \otimes_S R$ by (i) and $C_1 \otimes_S K \cong C \otimes_R K$. □

Therefore, we can successively investigate the composition algebras over affine schemes

$$\begin{aligned} Y - \{P_{i_0}\} &= \operatorname{Spec} S_0, \\ Y - \{P_{i_0}, P_{i_1}\} &= \operatorname{Spec} S_1, \\ &\vdots \\ Y - \{P_0, \dots, P_n\} &= \operatorname{Spec} S_n \end{aligned}$$

for $P_{i_0}, P_{i_1} \dots \in \{P_0, \dots, P_n\}$, or equivalently the composition algebras over $S_0 \subset S_1 \subset \dots \subset S_n$.

Unless stated otherwise, only curves of genus zero are considered from now on. For the rest of this section let R be a ring satisfying

$$X := \operatorname{Spec} R \cong X' - \{P_0, \dots, P_n\}.$$

It is obvious that R depends both on X' and the points $P_0, \dots, P_n \in X'$ chosen.

The remainder of this section, as well as the next one, almost exclusively deals with composition algebras over R that have no zero divisors. Composition algebras containing zero divisors will be investigated in section 4. Using the classification of composition algebras over curves of genus zero we obtain the following result.

2.7. Proposition. *Let C be a composition algebra without zero divisors over R. Then the following are equivalent.*

(i) *C is defined over k.*
(ii) *$C \otimes_R K$ is unramified at $P_0, \ldots, P_n \in X'$.*
(iii) *$C \otimes_R K$ is defined over k.*

Proof. (i) trivially implies (iii).

(iii) implies (ii): If $C \otimes_R K$ is defined over k, it is clearly unramified at all points $P \in X'$.

(ii) implies (i): Let $C \otimes_R K$ be unramified at $P_0, \ldots, P_n \in X'$. Since C is a composition algebra over $R, \mathcal{C} := \widetilde{C}$ is a composition algebra over $X = \operatorname{Spec} R$. Extend $\mathcal{C}$ to a quadratic alternative $\mathcal{O}_{X'}$-algebra $\mathcal{C}'$ such that $\mathcal{C}'_{P_i}$ is a selfdual $\mathcal{O}_{P_i,X'}$-order in $C' := C \otimes_R K$. Let $N_{\mathcal{C}'}$ denote the norm of $\mathcal{C}'$, then $N_{\mathcal{C}'}$ is nondegenerate, because $N_{\mathcal{C}'}|_X$ is the norm of $\mathcal{C}$ and $N_{\mathcal{C}',P_i}$ also is nondegenerate for all $0 \leq i \leq n$ (1.9). Therefore, $\mathcal{C}'$ is a composition algebra over X'. Due to 2.3 it is defined over k, splits or is isomorphic to a Cayley-Dickson doubling of $D \otimes \mathcal{O}_{X'}$. In the first case $\mathcal{C}' \cong C_0 \otimes \mathcal{O}_{X'}$ for a suitable composition algebra C_0 over k and therefore, $\mathcal{C} = \mathcal{C}'|_X \cong C_0 \otimes \mathcal{O}_X$. Thus, $C \cong \Gamma(X, \mathcal{C}) \cong C_0 \otimes_k R$ is defined over k. In case $\mathcal{C}'$ splits it contains a composition subalgebra isomorphic to $\mathcal{O}_{X'} \oplus \mathcal{O}_{X'}$. So $\mathcal{C} = \mathcal{C}'|_X$ contains a composition subalgebra isomorphic to $\mathcal{O}_X \oplus \mathcal{O}_X$ and $k \oplus k \cong \Gamma(X, \mathcal{O}_X \oplus \mathcal{O}_X)$ is isomorphic to a composition subalgebra of $C \cong \Gamma(X, \mathcal{C})$, that is, C splits. This, however, contradicts the assumption that C does not have zero divisors.

If $\mathcal{C}' \cong \operatorname{Cay}(D \otimes \mathcal{O}_{X'}, \mathcal{P}, N)$, the quaternion algebra $D \otimes \mathcal{O}_{X'}$ is a composition subalgebra of $\mathcal{C}'$ implying that $D \otimes \mathcal{O}_X$ is a composition subalgebra of $\mathcal{C}$ and therefore, $D \otimes R$ is a composition subalgebra of C. However, this implies that C contains zero divisors (2.2); again a contradiction to the assumption. □

2.8. Corollary. *For a composition algebra C without zero divisors over R the composition algebra $C \otimes K$ ramifies at most at $P_0, \ldots, P_n \in X'$.*

Proof. 2.5, 2.7. □

3. Composition subalgebras

Let Y be a curve over k and let $K := \kappa(Y)$ be its function field. Most of the proofs in this section rely on the following version of Hurwitz's theorem by van Geel.

3.1. Theorem ([G], 1.21)**.** *Let $\mathcal{L}$ be an $\mathcal{O}_Y$-lattice in a (nondegenerate) symmetric bilinear space V over K of dimension n. Then*

$$\chi(\mathcal{L}) = n(1 - g_Y) + \frac{1}{2} \deg_{\mathcal{O}_Y}[\mathcal{L}^{\#} : \mathcal{L}].$$

Consider the ring R with

$$\operatorname{Spec} R = Y - \{P_0, \ldots, P_n\}$$

for closed points $P_0, \ldots, P_n \in Y$ and a composition algebra C over R. For $C' := C \otimes_R K$, it is possible to calculate the Euler characteristic of certain maximal $\mathcal{O}_Y$-orders in C', which will be used repeatedly later on.

3.2. Lemma. *Let $\mathcal{C} := \widetilde{C}$ be the composition algebra over* $\operatorname{Spec} R$ *determined by C and let r be the rank of C. Assume* $\operatorname{char} k \neq 2$*, or that $r > 2$ and k is perfect. For every maximal $\mathcal{O}_Y$-order $\mathcal{C}'$ extending $\mathcal{C}$, i.e., $\mathcal{C}'|_X = \mathcal{C}$, the Euler characteristic is*

$$\chi(\mathcal{C}') = r\left(1 - g_Y - \frac{1}{4}\sum_{P \in \operatorname{Ram} C'} \deg P\right),$$

where $\operatorname{Ram} C' := \{P \in Y \mid C' \text{ ramifies at } P\}$.

Proof. C is a maximal R-order in C'. Since $\operatorname{Spec} R$ is an open, dense affine subscheme of Y, one can extend $\mathcal{C} = \widetilde{C}$ to a maximal $\mathcal{O}_Y$-order $\mathcal{C}'$ in C' by choosing a maximal $\mathcal{O}_{P_i,Y}$-order in C' for all $0 \leq i \leq n$ and gluing these (1.7). If C' is unramified at $P \in Y$, then the dual lattice satisfies $\mathcal{C}'^{\#}_P = \mathcal{C}'_P$ and

$$v_P([[\mathcal{C}'^{\#}_P : \mathcal{C}'_P] : \mathcal{O}_{P,Y}]) = v_P([\mathcal{O}_{P,Y} : \mathcal{O}_{P,Y}]) = 0.$$

If C' is not unramified at $P \in Y$, it ramifies separably due to the hypotheses on k, and $[\mathcal{C}'^{\#}_P : \mathcal{C}'_P] = m_P^{\frac{r}{2}}$ ([P1], 6.2). Then

$$\begin{aligned} v_P([[\mathcal{C}'^{\#}_P : \mathcal{C}'_P] : \mathcal{O}_{P,Y}]) &= v_P([\pi_P^{\frac{r}{2}}\mathcal{O}_{P,Y} : \mathcal{O}_{P,Y}]) \\ &= v_P([\mathcal{O}_{P,Y} : \pi_P^{\frac{r}{2}}\mathcal{O}_{P,Y}]^{-1}) = -\frac{r}{2}, \end{aligned}$$

with $m_P = (\pi_P)$ implies

$$\begin{aligned} \deg_{\mathcal{O}_Y}[\mathcal{C}'^{\#} : \mathcal{C}'] &= \sum_{P \in Y} v_P([[\mathcal{C}'^{\#}_P : \mathcal{C}'_P] : \mathcal{O}_{P,Y}]) \deg P \\ &= \sum_{P \in \operatorname{Spec} R} v_P([[\mathcal{C}'^{\#}_P : \mathcal{C}'_P] : \mathcal{O}_{P,Y}]) \deg P \\ &\quad + \sum_{i=0}^{n} v_{P_i}([[\mathcal{C}'^{\#}_{P_i} : \mathcal{C}'_{P_i}] : \mathcal{O}_{P_i,Y}]) \deg P_i \\ &= -\frac{r}{2}\sum_{i=0}^{n} m_i \deg P_i, \end{aligned}$$

where $m_i = 0$ (respectively, $m_i = 1$) in case C' is unramified at P_i (respectively, separably ramified at P_i). Hence, by 3.1

$$\begin{aligned} \chi(\mathcal{C}') &= r(1 - g_Y) - \frac{r}{4}\sum_{i=0}^{n} m_i \deg P_i \\ &= r\left(1 - g_Y - \frac{1}{4}\sum_{P \in \operatorname{Ram} C'} \deg P\right). \end{aligned}$$

□

Obviously, the Euler characteristic $\chi(\mathcal{C}')$ is independent of the maximal $\mathcal{O}_Y$-order $\mathcal{C}'$ in C' chosen in 3.2. It only depends on the rank of C, and on where C' ramifies.

$\mathcal{C}'$ is a quadratic alternative $\mathcal{O}_Y$-algebra with norm $N_{\mathcal{C}'} = \underline{N'}|_{\mathcal{C}'}$, N' denoting the norm of C'. It is a composition algebra if and only if C' is unramified at all $P \in Y$. To see this, note that if $\mathcal{C}'$ is a composition algebra over Y, then $\check{\mathcal{C}}'_P \cong \mathcal{C}'_P$ for all $P \in Y$. So $\mathcal{C}'_P$ is a selfdual $\mathcal{O}_{P,Y}$-order in C' which implies that C' is unramified at all $P \in Y$. Conversely, let C' be unramified at all $P \in Y$; hence there exists a selfdual $\mathcal{O}_{P,Y}$-order in C' for all $P \in Y$, so $\check{\mathcal{C}}'_P \cong \mathcal{C}'_P$ for all $P \in Y$ and $\mathcal{C}'$ is a composition algebra.

The Euler characteristic $\chi(\mathcal{C}') = h^0(X', \mathcal{C}') - h^1(X', \mathcal{C}')$ gives a lower bound for $h^0(X', \mathcal{C}')$ which becomes nontrivial in case $\chi(\mathcal{C}')$ is strictly greater than one. This will be used in the proof of 3.3.

From now on let X' denote a nonrational curve of genus zero over a base field k, where k is of $\operatorname{char} k \neq 2$, or perfect. Furthermore, let $K := \kappa(X')$ and let R be a ring such that

$$X := \operatorname{Spec} R = X' - \{P_0, \dots, P_n\}$$

for closed points $P_0, \dots, P_n \in X'$. Recall that a curve of genus zero contains k-rational points if and only if it is rational.

3.3. Theorem. (i) *Let $\sum_{i=0}^n \deg P_i = 2$. Then every composition algebra C without zero divisors of rank $r > 2$ over R contains a composition subalgebra of rank $\frac{r}{2}$ which is defined over k.*

(ii) *Let $\sum_{i=0}^n \deg P_i = 3$. Then every octonion algebra C without zero divisors over R contains a torus which is defined over k.*

Proof. (i) $C' := C \otimes_R K$ is a composition division algebra over K (1.2). Extend the composition algebra $\mathcal{C} := \widetilde{C}$ over X to a maximal $\mathcal{O}_{X'}$-order $\mathcal{C}'$ in C'. Then 3.2 implies

$$h^0(X', \mathcal{C}') \geq \chi(\mathcal{C}') = r\left(1 - \frac{1}{4}\sum_{P \in \operatorname{Ram} C'} \deg P\right) \geq r\left(1 - \frac{1}{4}\cdot 2\right) = \frac{r}{2}.$$

$\mathcal{C}'$ is a quadratic alternative $\mathcal{O}_{X'}$-algebra. Hence, $\Gamma(X', \mathcal{C}')$ is a finite-dimensional quadratic alternative k-subalgebra of the composition division algebra C' over K, and, by the hypotheses on k, $\Gamma(X', \mathcal{C}')$ is also a composition algebra. Furthermore, $\Gamma(X', \mathcal{C}') \otimes \mathcal{O}_{X'}$ is a composition subalgebra of $\mathcal{C}'$ ([P1], 5.2) of rank $\dim_k \Gamma(X', \mathcal{C}') = h^0(X', \mathcal{C}') \leq r$. Therefore, $r \geq \dim_k \Gamma(X', \mathcal{C}') \geq \frac{r}{2}$ and this implies $\dim_k \Gamma(X', \mathcal{C}') \in \{r, \frac{r}{2}\}$. Let $\sigma : X' \to \operatorname{Spec} k$ denote the structure morphism of X' and let $C_0 := \Gamma(X', \mathcal{C}')$. If $\dim_k C_0 = r$ it follows that

$$\mathcal{C}' \cong \sigma^* C_0 \cong C_0 \otimes \mathcal{O}_{X'}$$

is a composition algebra defined over k, and thus, that $\mathcal{C} = \mathcal{C}'|_X \cong C_0 \otimes \mathcal{O}_X$ also is defined over k implying $C \cong \Gamma(X, \mathcal{C}) \cong C_0 \otimes_k R$.

If $\dim_k C_0 = \frac{r}{2}$, it follows that $\sigma^* C_0 \cong C_0 \otimes \mathcal{O}_{X'}$ is a composition subalgebra of $\mathcal{C}'$ defined over k and of rank $\frac{r}{2}$. Then $\sigma^* C_0|_X \cong C_0 \otimes \mathcal{O}_X$ is a composition subalgebra of $\mathcal{C}'|_X = \mathcal{C}$ which is defined over k and of rank $\frac{r}{2}$, and $C_0 \otimes R \cong \Gamma(X, C_0 \otimes \mathcal{O}_X) \subset \Gamma(X, \mathcal{C}) \cong C$, that is, C contains a composition subalgebra of rank $\frac{r}{2}$ defined over k.

(ii) Using the same notation as in the proof of (i) we obtain the inequality

$$8 \geq h^0(X', \mathcal{C}') \geq \chi(\mathcal{C}') = 8\left(1 - \frac{1}{4}\sum_{P \in \operatorname{Ram} C'} \deg P\right) \geq 8\left(1 - \frac{1}{4}\cdot 3\right) = 2.$$

The proof is the same as for (i), only in this case the three possibilities $\dim_k \Gamma(X', \mathcal{C}') = 2, 4$ or 8 have to be discussed. $\square$

3.4. *Remark.* Let C be a composition algebra without zero divisors over R of rank r. Again let $\mathcal{C}'$ be an extension of $\mathcal{C} = \widetilde{C}$ to a maximal $\mathcal{O}_{X'}$-order in $C' = C \otimes_R K$, then the following are obviously equivalent.

(i) C' is unramified at $P_0, \dots, P_n \in X'$.
(ii) $\mathcal{C}'$ is a composition algebra over X'.
(iii) $\chi(\mathcal{C}') = r$.
(iv) C is defined over k.

3.5. Corollary. *In the situation of 3.3*(i) *every composition algebra C without zero divisors of rank $r > 2$ over R can be realized by a generalized Cayley-Dickson doubling of a composition algebra defined over k. That is, $C \cong \mathrm{Cay}(D_0 \otimes R, P, N)$ for some composition algebra D_0 over k, $P \in \mathrm{Pic}_r(D_0 \otimes R)$ of norm one and $N : P \to R$ a norm on P.*

Proof. [3.3(i), 1.5]. □

In case char $k \neq 2$ the ring R itself is a composition subalgebra of every torus C over R, so $C \cong \mathrm{Cay}(R, P, N)$ for a suitable $P \in \mathrm{Pic}(R)$ with $P \otimes P \cong R$.

3.6. Theorem. *Let C be a composition algebra over R without zero divisors and $C' := C \otimes K$.*

(i) *Suppose that there exists a $P_{i_0} \in \{P_0, \dots, P_n\}$ with $\deg P_{i_0} = 2$. If C has rank $r > 2$ and C' ramifies exactly at P_{i_0}, then C is the Cayley-Dickson doubling of a composition algebra which is defined over k.*
(ii) *Suppose that there exist two k-rational points $P_{i_0}, P_{i_1} \in \{P_0, \dots, P_n\}$. If C has rank $r > 2$ and C' ramifies exactly at P_{i_0} and P_{i_1}, then C is the Cayley-Dickson doubling of a composition algebra which is defined over k.*
(iii) *Suppose that there exists a $P_{i_0} \in \{P_0, \dots, P_n\}$ with $\deg P_{i_0} = 3$. If C has rank 8 and C' ramifies exactly at P_{i_0}, then C contains a torus which is defined over k.*
(iv) *Suppose that there exist $P_{i_0}, P_{i_1} \in \{P_0, \dots, P_n\}$ with $\deg P_{i_0} = 1$, $\deg P_{i_1} = 2$. If C has rank 8 and C' ramifies exactly at P_{i_0} and P_{i_1}, then C contains a torus which is defined over k.*
(v) *Suppose that there exist three k-rational $P_{i_0}, P_{i_1}, P_{i_2} \in \{P_0, \dots, P_n\}$. If C has rank 8 and C' ramifies exactly at P_{i_0}, P_{i_1} and P_{i_2}, then C contains a torus which is defined over k.*

In all the above cases C is not defined over k.

Proof. (i) Let C be a composition algebra without zero divisors of rank $r > 2$ over R and let C' ramify exactly at P_{i_0}. C is not defined over k (2.7). We can extend the composition algebra $\mathcal{C} = \widetilde{C}$ over X to a composition algebra $\mathcal{C}_1$ over $X_1 := \mathrm{Spec}\, S \cong X' - \{P_{i_0}\}$ such that $\mathcal{C}_{1,P_i}$ is a maximal $\mathcal{O}_{P_i,X_1}$-order in C' for all i, $0 \leq i \leq n$, $i \neq i_0$. So $\mathcal{C}_1$ is a maximal $\mathcal{O}_{X_1}$-order in C'. There exists a composition algebra C_1 over S with $\mathcal{C}_1 = \widetilde{C}_1$ (1.4), and C_1 contains a composition subalgebra $D_0 \otimes_k R$ of rank $\frac{r}{2}$ (3.3)(i). Therefore, $D_0 \otimes \mathcal{O}_{X_1}$ is a composition subalgebra of $\mathcal{C}_1$ and $(D_0 \otimes \mathcal{C}_{X_1})|_X \cong D_0 \otimes \mathcal{O}_X$ is a composition subalgebra of $\mathcal{C}_1|_X = \widetilde{C}$. This implies that $D_0 \otimes R$ is a composition subalgebra of C.

(ii) is proved analogously using 3.4(i), while (iii), (iv) and (v) follow using 3.3(ii). □

3.7. *Remark.* (a) By Petersson [P1], 6.8 every composition algebra of rank r over the polynomial ring $k[t]$ (with $r > 2$ in case $\operatorname{char} k = 2$) is defined over k.

So whenever there exists a k-rational point $P_{i_0} \in \{P_0, \dots, P_n\}$, then the same argument as in 3.6 shows that $C' = C \otimes_R K$ does not ramify exactly at P_{i_0} for any composition algebra C over R (of rank $r > 2$ in case $\operatorname{char} k = 2$).

(b) To prove 3.6 we can also employ the technique applied in the proof of 3.3. In particular, this approach shows that, given a maximal $\mathcal{O}_{X'}$-order in C' satisfying $\mathcal{C}'|_X = \widetilde{C}$, the algebra $\Gamma(X', \mathcal{C}') \otimes_k R$ always turns out to be a composition subalgebra of C.

It can now be shown that the composition subalgebras whose existence was proved in 3.3 and 3.6 are uniquely determined up to isomorphism unless the composition algebras under consideration are defined over k. This has obvious consequences for classification purposes later on.

3.8. Theorem. *Let C be a composition algebra without zero divisors of rank r, $r > 2$ for $\operatorname{char} k = 2$, over R and let $\mathcal{C}'$ be a maximal $\mathcal{O}_{X'}$-order in $C' = C \otimes K$ such that $\mathcal{C}'|_X = \widetilde{C}$. If $\operatorname{char} k \neq 2$, or $\operatorname{char} k = 2$ and $\dim_k \Gamma(X', \mathcal{C}') \geq 2$, then $\Gamma(X', \mathcal{C}') \otimes_k R$ is, up to isomorphism, the only composition subalgebra of C of rank $s := \dim_k \Gamma(X', \mathcal{C}')$ which is defined over k.*

If C is defined over k, then $s = \operatorname{rank} C$ and $C \cong \Gamma(X', \mathcal{C}') \otimes_k R$.

Proof. $\mathcal{C}'$ is a quadratic alternative $\mathcal{O}_{X'}$-algebra (3.2) and thus, $\Gamma(X', \mathcal{C}')$ is a finite-dimensional quadratic alternative k-subalgebra of the composition division algebra C' over K. By the above hypotheses it is a composition algebra over k, and furthermore, $\Gamma(X', \mathcal{C}') \otimes \mathcal{O}_{X'}$ is a composition subalgebra of $\mathcal{C}'$ of rank $s \leq r$. This implies that $\Gamma(X', \mathcal{C}') \otimes R$ is a composition subalgebra of C of rank s (cf. 3.3).

Assume that D_0 is a composition algebra over k, $\dim_k D_0 = s \leq r$, such that $D_0 \otimes R$ is a composition subalgebra of C. Extend the composition subalgebra $D_0 \otimes \mathcal{O}_X$ of $\widetilde{C}$ to an $\mathcal{O}_{X'}$-algebra $\mathcal{D}'$ such that $\mathcal{D}'_{P_i} = \Gamma(X', \mathcal{C}') \otimes \mathcal{O}_{P_i, X'} \cong (\Gamma(X', \mathcal{C}') \otimes \mathcal{C}_{X'})_{P_i}$ for all i, $0 \leq i \leq n$. Then $\mathcal{D}'$ is a composition subalgebra of $\mathcal{C}'$ because $\mathcal{D}'|_X = D_0 \otimes \mathcal{O}_X \subset \mathcal{C}'|_X$ and $\mathcal{D}'_{P_i} \subset \mathcal{C}'_{P_i}$, for all i, $0 \leq i \leq n$.

$\mathcal{D}'$ either splits, or is defined over k, or (for nonrational X' with associated quaternion division algebra D) is a Cayley-Dickson doubling of $D \otimes \mathcal{O}_{X'}$ (2.3).

In the first case $\mathcal{D}'$ contains a composition subalgebra isomorphic to $\mathcal{O}_{X'} \oplus \mathcal{O}_{X'}$ implying that $\mathcal{D}'_\xi \subset C'$ contains a composition subalgebra isomorphic to $\mathcal{O}_{\xi, X'} \oplus \mathcal{O}_{\xi, X'} \cong K \oplus K$ and contradicting the fact that C' is a division algebra.

If X' is nonrational and $\mathcal{D}'$ a Cayley-Dickson doubling of $D \otimes \mathcal{O}_{X'}$, this implies that $D_0 \otimes \mathcal{O}_X \subset \widetilde{C}$ is a composition subalgebra and thus, $D \otimes R$ is a composition subalgebra of C. But $D \otimes R$ has zero divisors (2.2) contradicting the fact that C' does not.

Thus, $\mathcal{D}' = D_1 \otimes \mathcal{O}_{X'}$ for a composition algebra D_1 over k with $\dim_k D_1 = s$ and $\mathcal{D}'|_X = D_1 \otimes \mathcal{O}_X \cong D_0 \otimes \mathcal{O}_X$ implying $D_0 \otimes R \cong D_1 \otimes R$. Moreover, $\mathcal{D}' \subset \mathcal{C}'$, $D_1 \cong \Gamma(X', \mathcal{D}') \subset \Gamma(X', \mathcal{C}')$ is a composition subalgebra and comparing dimensions shows $D_1 \cong \Gamma(X', \mathcal{C}')$. It follows that $D_0 \otimes R \cong \Gamma(X', \mathcal{C}') \otimes R$. In particular, whenever C is defined over k and $\dim_k \Gamma(X', \mathcal{C}') = r$ (3.4) it follows that $C \cong \Gamma(X', \mathcal{C}') \otimes R$. □

3.9. Corollary. (a) *In the situation of 3.6*(i) (*resp. 3.6*(ii)) *the composition algebra C is the Cayley-Dickson doubling of a composition algebra defined over k which is uniquely determined up to isomorphism.*

(b) *In the situation of 3.6*(iii) (*resp. 3.6*(iv), (v)) *the composition algebra C contains a torus defined over k which is uniquely determined up to isomorphism.*

Proof. [3.6, 3.7(b), 3.8]. □

In particular, 3.9(a) implies the following: Consider the Cayley-Dickson doublings $\mathrm{Cay}(C_0, P_0, N_0)$, $\mathrm{Cay}(C_1, P_1, N_1)$ of two composition algebras C_0, C_1 over R of rank $s \in \{2, 4\}$ which are defined over k. If these Cayley-Dickson doublings are algebras without zero divisors ramifying over K exactly at a point $P_{i_0} \in \{P_0, \dots, P_n\}$ of $\deg P_{i_0} = 2$ as in 3.6(i), then $\mathrm{Cay}(C_0, P_0, N_0)$ and $\mathrm{Cay}(C_1, P_1, N_1)$ are not isomorphic unless $C_0 \cong C_1$. The same conclusion holds if the Cayley-Dickson doublings are without zero divisors and ramify exactly at two k-rational points $P_{i_0}, P_{i_1} \in \{P_0, \dots, P_n\}$ as in 3.6(ii).

3.10. Corollary. *Let C be a composition algebra without zero divisors of rank $r > 2$ over R, where R is one of the following rings*:

(i) $R = \{\frac{g(t)}{f(t)^j} \in k(t) \mid j \geq 0,\ g(t) \in k[t]$ *of* $\deg g \leq 2j\}$ *where $f \in k[t]$ is a monic irreducible polynomial of degree two*;

(ii) $R = k[t, \frac{1}{t}]$*, the ring of Laurent polynomials*;

(iii) $R = k[t, \sqrt{at^2 + b}]$ *with k a field of characteristic not two and $a, b \in k^\times$ such that $(a, b)_k$ is a quaternion division algebra.*

If C is not defined over k, then C is the Cayley-Dickson doubling of a composition algebra defined over k which is uniquely determined up to isomorphism.

Proof. (i) Let R be a ring such that $\mathrm{Spec}\, R = \mathbf{P}^1_k - \{P_0\}$ for $P_0 \in \mathbf{P}^1_k = \mathrm{Proj}\, k[x_0, x_1]$ with $\deg P_0 = 2$. P_0 is represented by the principal ideal generated by an irreducible homogeneous polynomial $f_h(x_0, x_1) \in k[x_0, x_1]$ of degree two and $\mathbf{P}^1_k - \{P_0\} \cong \mathrm{Spec}(k[x_0, x_1]_{f_h})$ by [H], II.2.5a). A straightforward verification shows that $R = \{\frac{g(t)}{f(t)^j} \in k(t) \mid j \geq 0,\ g(t) \in k[t] \text{ of } \deg g(t) \leq 2j\}$ for $t = \frac{x_0}{x_1}$, where we may assume that $f(t) := f_h(t, 1) \in k[t]$ is a monic and irreducible polynomial of degree two. The assertion now follows from 3.9(a).

(ii) Let R be a ring such that $\mathrm{Spec}\, R = \mathbf{P}^1_k - \{P_0, P_1\}$ for two k-rational points $P_0, P_1 \in \mathbf{P}^1_k$. Since $\mathbf{P}^1_k = \mathbf{A}^1_k \cup \infty$ with $\mathbf{A}^1_k = \mathrm{Spec}\, k[t]$, assume w.l.o.g. $P_0 = \infty$, $P_1 = (t)$, then $\mathbf{P}^1_k - \{P_0, P_1\} = \mathbf{A}^1_k - \{(t)\} = \mathrm{Spec}\, k[t, \frac{1}{t}]$. Again 3.9(a) yields the assertion.

(iii) Let R be a ring such that $\mathrm{Spec}\, R = X' = \{P_0\}$ for $P_0 \in X'$ of $\deg P_0 = 2$, $\kappa(P_0) = k(\sqrt{a})$, and for a nonrational curve X' over a field k of $\mathrm{char}\, k \neq 2$ with associated quaternion division algebra $(a, b)_k$. It may be assumed that P_0 corresponds with the unique extension of the place ∞ of $k(t)$ to $K = \kappa(X') = k(t, \sqrt{at^2 + b})$ (cf. [Pf], p. 259). Then $R = k[t, \sqrt{at^2 + b}]$ by a simple calculation and the proof follows from 3.9(a). □

3.11. Corollary. *Let C be an octonion algebra without zero divisors over R, where R is one of the following rings*:

(i) $R = \{\frac{g(t)}{f(t)^j} \in k(t) \mid j \geq 0,\ g(t) \in k[t]$ *of* $\deg g \leq 3j\}$*, where $f(t) \in k[t]$ is a monic irreducible polynomial of* $\deg f = 3$.

(ii) $R = k[t, \frac{1}{f(t)}]$*, where $f(t) \in k[t]$ is a monic irreducible polynomial of* $\deg f = 2$.

(iii) $R = k[t, \frac{1}{t}, \frac{1}{t-b}]$*, where $b \in k^\times$.*

Then C contains a torus which is defined over k. If C itself is not defined over k, then this torus is uniquely determined up to isomorphism.

Proof. (i) Let R be a ring such that $\operatorname{Spec} R = \mathbf{P}^1_k - \{P_0\}$ for $P_0 \in \mathbf{P}^1_k = \operatorname{Proj} k[x_0, x_1]$ of $\deg P_0 = 3$. As in the proof of 3.10(i) it follows that $R = \{\frac{g(t)}{f(t)^j} \in k(t) \mid j \geq 0,$ $g(t) \in k[t]$ of $\deg g \leq 3j\}$ for $f(t) \in k[t]$ a monic and irreducible polynomial of degree three representing P_0. The assertion now is a consequence of 3.9(b).

(ii) Let R be a ring such that $\operatorname{Spec} R = \mathbf{P}^1_k - \{P_0, P_1\}$ for $P_0, P_1 \in \mathbf{P}^1_k$ of $\deg P_0 = 1$, $\deg P_1 = 2$. Assume $P_0 = \infty$, then P_1 is represented by the principal ideal generated by a monic irreducible polynomial $f(t) \in k[t]$ of $\deg f = 2$ and $\mathbf{P}^1_k - \{P_0, P_1\} = \mathbf{A}^1_k - \{P_0\} = \operatorname{Spec} k[t, \frac{1}{f(t)}]$. Now 3.9(b) yields the assertion.

(iii) Let R be a ring such that $\operatorname{Spec} R = \mathbf{P}^1_k - \{P_0, P_1, P_2\}$ for three k-rational points $P_0, P_1, P_2 \in \mathbf{P}^1_k$. Assume $P_0 = \infty$, $P_1 = (t)$, $P_2 = (t-b)$ for a $b \in k^\times$, then $R = k[t, \frac{1}{t}, \frac{1}{t-b}]$ and 3.9(b) implies the assertion. □

4. Composition algebras with zero divisors

Split composition algebras over locally ringed spaces (1.2) were investigated by Petersson ([P1], 2.7, 3.5). Applied to affine schemes $\operatorname{Spec} R$, his results can easily be conveyed to split composition algebras over R recalling the equivalence of these categories (1.4).

However, if the underlying ring is a Dedekind domain, the split octonion algebras turn out to be much simpler than in the general case.

4.1. Proposition. *Let S be a Dedekind domain. Then the vector matrix algebra* $\operatorname{Zor} S$ *is (up to isomorphism) the only split octonion algebra over S.*

Proof. Any split octonion algebra over a ring S is isomorphic to $\operatorname{Zor}(T, \alpha)$ where T is a projective S-module of rank 3 with $\det T \cong S$ and $\alpha : \det T \xrightarrow{\sim} S$ an isomorphism ([P1], 3.5). Since S is a Dedekind domain, we have $T \cong S^2 \oplus L$ for some $L \in \operatorname{Pic} S$ (see for instance [BJ], p. 136) and because of $\det T \cong S \otimes S \otimes L \cong L$ the only such T is $T \cong S^3$.

The isomorphism $\alpha : \det T \xrightarrow{\sim} S$ is uniquely determined up to a scalar $\nu \in S^\times$. Choose $\alpha_1 : \det T \xrightarrow{\sim} S$, $\alpha_1(u_1 \wedge u_2 \wedge u_3) = \det(u_1, u_2, u_3)$. Then α_1 induces a bilinear map $\times_{\alpha_1} : T \times T \to \check{T}$, $u \times_{\alpha_1} v := \alpha(u \wedge v \wedge \_)$. Here, $u_1 \times_{\alpha_1} u_2 = u_1 \times u_2$ is the usual vector product and $\operatorname{Zor}(T, \alpha_1) \cong \operatorname{Zor}(S^3, \alpha_1) \cong \operatorname{Zor} S$. Given an arbitrary $\alpha : \det T \xrightarrow{\sim} S$ there is a $\nu \in S^\times$ such that $\alpha(u_1 \wedge u_2 \wedge u_3) = \nu \det(u_1, u_2, u_3) = \langle u_3, \nu(u_1 \times u_2)\rangle$.

$\langle u, v\rangle := u^t v$ is nondegenerate and thus, the bilinear map $\times_\alpha : T \times T \to \check{T}$ induced by α satisfies $u_1 \times_\alpha u_2 = \nu(u_1 \times u_2)$. It follows that $(R^3, \times_\alpha) \xrightarrow{\sim} (S^3, \times)$, $z \to \nu z$ is an algebra isomorphism and $\varphi : (S^3, \alpha) \to (S^3, \alpha_1)$, $\varphi((u_1, u_2, u_3)^t) = (\nu u_1, u_2, u_3)^t$ an isomorphism satisfying $\alpha_1 \circ (\det \varphi) = \alpha$. This implies $\operatorname{Zor}(S^3, \alpha) \cong \operatorname{Zor}(S^3, \alpha_1)$ by [P1], 3.4. □

For the remainder of this section let k be a field of $\operatorname{char} k \neq 2$. Let φ denote the norm of $D = (a, b)_k$, the associated quaternion division algebra for a nonrational curve X' of genus zero. It is well-known that $\ker(W(k) \to W(K)) = \varphi W(k)$ (see, for instance, [S], 4.5.4). The following is the result of a straightforward calculation.

4.2. Proposition. *Let R be an integral domain with $k \subset R$ such that* $\operatorname{Quot}(R) =$ *K, where K is the function field of a nonrational curve X'. Let C_0 be a composition algebra over k. $C_0 \otimes R$ has zero divisors if and only if one of the following holds.*

(i) *C_0 splits over k.*

(ii) *$(a, b)_k$ is isomorphic to a composition subalgebra of C_0.*

Proof. The conditions are clearly sufficient.

Conversely, let C_0 be a composition algebra over k with norm N and let $C_0 \otimes R$ have zero divisors. Then $C_0 \otimes R$ splits and $N \in \ker(W(k) \to W(K))$. If $\dim_k C_0 = 8$, then $n \cong \langle u_0, u_1 \rangle \otimes \varphi$ with $u_0, u_1 \in k^\times$, and we may assume $u_0 = 1$. Therefore $N \cong \varphi \perp u_1 \varphi$ is isometric to the norm of $\mathrm{Cay}((a,b)_k, -u_1)$ and $C_0 \cong \mathrm{Cay}((a,b)_k, -u_1)$. A similar argument for $\dim_k C_0 < 8$ completes the proof. □

Using Petersson's classification theorem for composition algebras over curves of genus zero (2.3) we can now characterize the composition algebras with zero divisors over the rings considered in this paper.

Therefore, again let R denote a ring such that

$$\mathrm{Spec}\, R = X' - \{P_0, \dots, P_n\}$$

for closed points $P_0, \dots, P_n \in X'$, X' a curve a genus zero over k ($\mathrm{char}\, k \neq 2$ here).

4.3. Theorem. *A composition algebra C over R has zero divisors if and only if one of the following holds.*

(i) *C is split, and thus isomorphic to $R \oplus R$, or to $\mathrm{End}_R(R \oplus L)$ for some $L \in \mathrm{Pic}\, R$, or to $\mathrm{Zor}\, R$.*

(ii) *X' is nonrational and $(a,b)_k \otimes R$ is isomorphic to a composition subalgebra of C.*

Proof. For a composition algebra C over R which has zero divisors, $C' := C \otimes_R K$ splits and therefore contains a selfdual $\mathcal{O}_{P_i,X'}$-order for all i, $0 \leq i \leq n$ ([Kü], 3.2.2 or [P1], 6.3). Extend $\mathcal{C} := \widetilde{C}$ to a quadratic alternative $\mathcal{O}_{X'}$-algebra such that $\mathcal{C}'_{P_i}$ is a selfdual $\mathcal{O}_{P_i,X'}$-order in C', $\mathcal{C}'$ is a composition algebra over X' (3.2) and so either defined over k, or split of rank ≥ 4, or $\mathcal{C}' = \mathrm{Cay}((a,b)_k, \mathcal{P}, N)$ by 2.3.

If $\mathcal{C}' \cong C_0 \otimes \mathcal{O}_{X'}$ for a composition algebra C_0 over k, then $C \cong \Gamma(X, \mathcal{C}') = C_0 \otimes_k R$. For rational $X' = \mathbf{P}^1_k$ and $K = k(t)$, obviously C' has zero divisors if and only if C_0 does, that is, C_0 is a split composition algebra over k. For nonrational X', it follows that C_0 either splits or contains a composition subalgebra isomorphic to $(a,b)_k$ (4.2).

If $\mathcal{C}'$ splits, it contains a composition subalgebra isomorphic to $\mathcal{O}_{X'} \oplus \mathcal{O}_{X'}$ and this immediately implies that $C \cong \Gamma(X, \mathcal{C})$ contains a composition subalgebra isomorphic to $R \oplus R$; hence C also splits.

If $\mathcal{C}' \cong \mathrm{Cay}((a,b)_k \otimes \mathcal{O}_{X'}, \mathcal{P}, N)$, then again $C \cong \Gamma(X, \mathcal{C})$ contains $(a,b)_k \otimes R \cong \Gamma(X, (a,b)_k \otimes \mathcal{O}_{X'})$ as a composition subalgebra. This concludes the proof. □

When the u-invariant of the function field K is shown and $u(K) \leq 6$, every octonion algebra over R has zero divisors.

4.4. Example. (a) Let k be algebraically closed and $X' = \mathbf{P}^1_k$. Then $u(K) \leq 2$ ([S], 2.15.3), and every composition algebra over R of rank > 2 splits.

(b) Let $k = \mathbf{F}_q$ be a finite field, $q = p^n$ with $p \neq 2$. Then $u(K) = 4$ for $X' = \mathbf{P}^1_k$ and $\mathrm{Zor}\, R$ is the only octonion algebra over R. The same holds for any field k of transcendence degree one over an algebraically closed field, because in this case also $u(K) \leq 4$.

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